Solution for MS&E 322

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October 6, 2025

1 Assignment 1

(1) (a)

By stationary increment property of $B(\cdot)$, the distribution of $B(s)|\{B(r)=a, B(t)=b\}$ is equivalent to the distribution of $a+B(s-r)|\{B(t-r)=b-a\}$. By orthogonal decomposition:

$$\begin{aligned} B(s-r) &= \frac{\mathrm{Cov}(B(s-r), B(t-r))}{\mathrm{Var}(B(t-r))} B(t-r) + Z \\ &= \frac{s-r}{t-r} B(t-r) + Z, \end{aligned}$$

where $Z \perp\!\!\!\perp B(t-r)$ with $\mathbb{E}[Z]=0$. The variance of Z satisfies:

$$\begin{aligned} \mathbb{E}[Z^2] &= \mathbb{E}\left[B(s-r) - \frac{s-r}{t-r} B(t-r)\right]^2 \\ &= (s-r) - 2\frac{(s-r)^2}{t-r} + \frac{(s-r)^2}{t-r} \\ &= \frac{(s-r)(t-s)}{t-r} \end{aligned}$$

Thus, we have:

$$\begin{aligned} \mathbb{E}[B(s-r)|\{B(t-r)=b-a\}] &= \frac{s-r}{t-r}(b-a) \\ \mathbb{E}[B(s-r)^2|\{B(t-r)=b-a\}] &= \left(\frac{s-r}{t-r}(b-a)\right)^2 + \frac{(s-r)(t-s)}{t-r}. \end{aligned}$$

In conclusin,

$$\begin{aligned} \mathbb{E}[B(s)|\{B(r)=a, B(t)=b\}] &= a + \mathbb{E}[B(s-r)|\{B(t-r)=b-a\}] \\ &= a + \frac{s-r}{t-r}(b-a) \\ \mathrm{Var}[B(s)|\{B(r)=a, B(t)=b\}] &= \mathbb{E}[B(s-r)^2|\{B(t-r)=b-a\}] - \mathbb{E}[B(s)|\{B(r)=a, B(t)=b\}]^2 \end{aligned}$$

$$= \frac{(s-r)(t-s)}{t-r}.$$

For a joint Gaussian distribution (X, Y), the orthogonal decomposition is:

$$X = \mathbb{E}[X|Y] + Z,$$

where $\sigma(Z) \perp\!\!\!\perp \sigma(Y)$. And there must exists $r \in \mathbb{R}$, such that

$$\mathbb{E}[X|Y] = rY.$$

Thus, multiplying $\mathrm{Cov}(\cdot, Y)$ both sides, we have:

$$\mathrm{Cov}(X, Y) = r\mathrm{Cov}(Y, Y),$$

which implies:

$$r = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(Y)}.$$

(1) (b)

For any fixed i, n, we set $r = \frac{2i}{2^n}$, $s = \frac{2i+1}{2^n}$, and $t = \frac{2i+2}{2^n}$. By (1) (a), we can generate $B(\frac{2i+1}{2^n})$ via:

$$\mathcal{N}\left(\frac{B\left(\frac{2i}{2^n}\right) + B\left(\frac{2i+2}{2^n}\right)}{2}, \frac{1}{2^{n+1}}\right).$$

(2)

On interval $\left[\frac{2i}{2^n}, \frac{2i+2}{2^n}\right]$, the maximum value of $|B_n(t) - B_{n-1}(t)|$ is achieved at middle point $\frac{2i+1}{2^n}$, which is:

$$\max_{t \in \left[\frac{2i}{2^n}, \frac{2i+2}{2^n}\right]} |B_n(t) - B_{n-1}(t)| = \left| \frac{B\left(\frac{2i}{2^n}\right) + B\left(\frac{2i+2}{2^n}\right)}{2} - B\left(\frac{2i+1}{2^n}\right) \right| := |X_{i,n}|.$$

By (1) (b), we have $X_{i,n} \sim \mathcal{N}\left(0, \frac{1}{2^{n+1}}\right)$. Thus,

$$\max_{t \in [0,1]} |B_n(t) - B_{n-1}(t)| = \max_{1 \le i \le 2^{n-1}} |X_{i,n}|.$$

(3)

By (2), we have:

$$\begin{aligned}\sum_{n \ge 1} \mathbb{P}\left(\max_{t \in [0,1]} |B_n(t) - B_{n-1}(t)| > \varepsilon\right) &= \sum_{n \ge 1} \mathbb{P}\left(\max_{1 \le i \le 2^{n-1}} |X_{i,n}| > \varepsilon\right) \\ &\le \sum_{n \ge 1} 2^{n-1} \mathbb{P}\left(|X_{1,n}| > \varepsilon\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n\geq 1} 2^n \exp\left(-2^n \varepsilon^2\right) \\
&< +\infty.
\end{aligned}
$$

By Borel-Cantelli lemma, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\omega \in \Omega_0 : \max_{t\in[0,1]} |B_n(t) - B_{n-1}(t)| > \varepsilon,\ \text{i.o.}\right) = 0.$$

However, this is not sufficient to prove the sequence $B_n(\cdot)$ is Cauchy. Instead, we set $\varepsilon = \sqrt{\frac{2n\ln 2}{2^n}}$:

$$
\begin{aligned}
\sum_{n\geq 1} \mathbb{P}\left(\max_{t\in[0,1]} |B_n(t) - B_{n-1}(t)| > \sqrt{\frac{2n\ln 2}{2^n}}\right) &= \sum_{n\geq 1} \mathbb{P}\left(\max_{1\leq i\leq 2^{n-1}} |X_{i,n}| > \sqrt{\frac{2n\ln 2}{2^n}}\right) \\
&\leq \sum_{n\geq 1} 2^{n-1} \mathbb{P}\left(|X_{1,n}| > \sqrt{\frac{2n\ln 2}{2^n}}\right) \\
&\leq \sum_{n\geq 1} 2^n \exp\left(-2n\ln 2\right) \\
&= \sum_{n\geq 1} \exp\left(-n\ln 2\right) \\
&< +\infty.
\end{aligned}
$$

By Borel-Cantelli lemma, we have:

$$\mathbb{P}\left(\omega \in \Omega_0 : \max_{t\in[0,1]} |B_n(t) - B_{n-1}(t)| > \sqrt{\frac{2n\ln 2}{2^n}},\ \text{i.o.}\right) = 0.$$

Thus, for any $\omega \in \Omega_0^c$, there exists $N(\omega)$, for any $n > N(\omega)$, we have:

$$\max_{t\in[0,1]} |B_n(t)(\omega) - B_{n-1}(t)(\omega)| < \sqrt{\frac{2n\ln 2}{2^n}}.$$

As $\sum_{n\geq 1} \sqrt{\frac{2n\ln 2}{2^n}}$ is summable, thus $B_n(\cdot)$ is a Cauchy sequence almost surely.

(4)

We should expect:

$$\hat{\mathbb{E}} S_{mid}(n) - \hat{\mathbb{E}} S_{left}(n) \approx 1/2,$$

when n is large and $\hat{\mathbb{E}}$ is the empirical mean among independent replications.