Assignment 3

00103335: Deep Learning and Reinforcement Learning

Fall 2026

Problem 1. Let $\lambda > 0$, let $(\mathcal{F}_t)_{t\geq 1}$ be a filtration, and let δ_1 be $\mathcal{F}_1$-measurable with $\mathbb{E}[|\delta_1|^4] < \infty$. Consider the scalar stochastic recursion

$$\delta_{t+1} = (1 - \lambda\eta_t)\delta_t + \eta_t\xi_t, \qquad t = 1, 2, \ldots, \tag{1}$$

where δ_t is $\mathcal{F}_t$-measurable and the noise satisfies

$$\mathbb{E}[\xi_t \mid \mathcal{F}_t] = 0, \qquad \mathbb{E}[|\xi_t|^4 \mid \mathcal{F}_t] \leq \sigma^4 \quad \text{almost surely} \tag{2}$$

for some $0 \leq \sigma < \infty$. Let the deterministic step sizes be

$$\eta_t = \frac{a}{(t + t_0)^\alpha}, \qquad \frac{1}{2} < \alpha \leq 1, \tag{3}$$

where $a > 0$, $t_0 \geq 0$ is chosen so that $0 < \lambda\eta_t \leq 1$, and, when $\alpha = 1$, $2\lambda a > 1$. This is the error recursion for one-dimensional SGD on a quadratic whose minimizer is zero.

(a) Prove that for every $c \in (0, 4\lambda)$, there are constants $C_0 < \infty$ and $t_1 \geq 1$ such that, for all $t \geq t_1$,

$$\mathbb{E}[|\delta_{t+1}|^4] \leq (1 - c\eta_t)\mathbb{E}[|\delta_t|^4] + C_0\eta_t^3.$$

(b) Prove that there is a constant $C < \infty$, independent of t, such that

$$\mathbb{E}[|\delta_t|^4] \leq C\eta_t^2.$$

(c) Prove that

$$\delta_t = O_{\mathbb{P}}(\sqrt{\eta_t}).$$

(d) Prove that

$$\delta_t \longrightarrow 0 \qquad \text{almost surely.}$$

Problem 2. Let $(\mathcal{F}_t)_{t\geq 1}$ be a filtration. Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be convex and differentiable, and suppose it attains its minimum at $\theta^\star$. Consider

$$\theta_{t+1} = \theta_t - \eta_t g_t, \tag{4}$$

where θ_t is $\mathcal{F}_t$-measurable and g_t is $\mathcal{F}_{t+1}$-measurable. Assume

$$\mathbb{E}[g_t \mid \mathcal{F}_t] = \nabla\Phi(\theta_t), \qquad \mathbb{E}[\|g_t\|^2 \mid \mathcal{F}_t] \leq G^2 \quad \text{almost surely,} \tag{5}$$

for some $0 \leq G < \infty$. Assume $\mathbb{E}\,\|\theta_1 - \theta^\star\|^2 \leq R^2$ for some $0 \leq R < \infty$. For deterministic step sizes $\eta_t > 0$, define

$$S_T := \sum_{t=1}^T \eta_t, \qquad Q_T := \sum_{t=1}^T \eta_t^2, \qquad \overline{\theta}_T^{(\eta)} := \frac{1}{S_T}\sum_{t=1}^T \eta_t\theta_t.$$

(a) Prove

$$\mathbb{E}[\|\theta_{t+1} - \theta^\star\|^2 \mid \mathcal{F}_t] \leq \|\theta_t - \theta^\star\|^2 - 2\eta_t\{\Phi(\theta_t) - \Phi(\theta^\star)\} + \eta_t^2 G^2.$$

(b) Prove

$$\mathbb{E}[\Phi(\overline{\theta}_T^{(\eta)})] - \Phi(\theta^\star) \leq \frac{R^2 + G^2 Q_T}{2S_T}. \tag{6}$$

(c) Assume $R > 0$ and $G > 0$. Set $\eta_t \equiv \eta := c/\sqrt{T}$ and define $\overline{\theta}_T := T^{-1}\sum_{t=1}^T \theta_t$. Determine $c > 0$ such that

$$\mathbb{E}[\Phi(\overline{\theta}_T)] - \Phi(\theta^\star) \leq \frac{RG}{\sqrt{T}}.$$